Relations

general mathematical framework for specifying relationships between pairs of objects.

"relation R on set A"

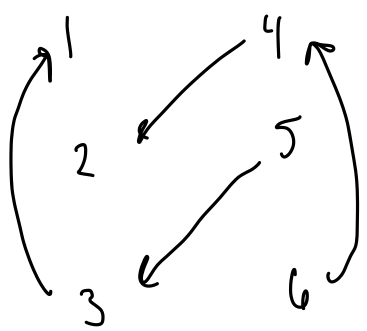
elements of A can be any object

Let $A = \{1,2,3,4,5,6\}$

Let R be a relation on A and let xRy iff $x = y + 2$.

→ one way we can define a relation

$3R1$, $4R2$, $5R3$, $6R4$ – relationships that exist on A



(a directed graph)

Properties of relations:

① Reflexivity

a) reflexive

iff every object is related to itself

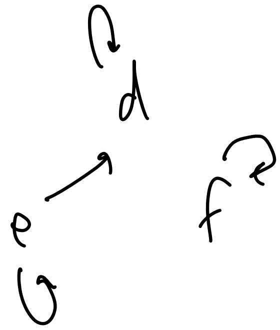
b) irreflexive

iff no object is related to itself

c) neither

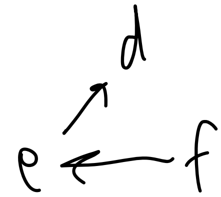
some objects related to themselves, some aren't

relation R on $\{d, e, f\}$



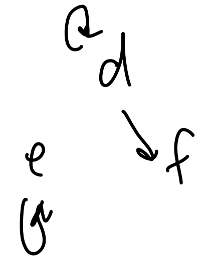
dRd, eRe, fRf, eRd

reflexive



eRd, fRe

irreflexive



dRd, eRe, dRf

neither

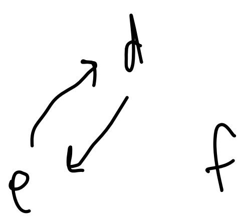
② Symmetry

a) symmetric

for all $x, y \in A$

if $xRy \to yRx$

F



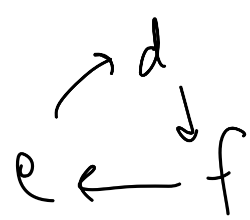
dRe, eRd

b) antisymmetric

for all $x, y \in A$

if $xRy \to y \cancel{R} x$

F



dRf, fRe, eRd

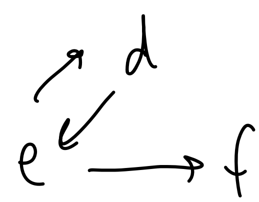
$\boxed{\text{d) both}}$

c) neither

for some $x, y \in A$

$xRy \to yRx$

$xRy \to y \cancel{R} x$



dRe, eRd, eRf

③ Transitivity

a) transitive

for all $x, y, z \in A$

if xRy, yRz, then xRz

b) not transitive / intransitive

exists $x, y, z \in A$

xRy, yRz and $x \cancel{R} z$

$$\begin{tikzcd} & d \arrow[dr] & \\ e \arrow[ur] \arrow[rr] & & f \end{tikzcd} \qquad \begin{tikzcd} & d \arrow[dr] & \\ e \arrow[ur] & & f \arrow[ll] \end{tikzcd}$$

$eRd, dRf \rightarrow eRf$ $\qquad$ $eRd, dRf, e\not{R}f$

★ note: $\begin{tikzcd} d \arrow[loop left] \arrow[r, bend left] & e \arrow[loop above] \arrow[l, bend left] \end{tikzcd}$ with self loops, it is. ~~not~~ transitive!

x, y, z from defn do not have to be unique

$\underset{x}{d}R\underset{y}{e}, \underset{y}{e}R\underset{z}{d} \rightarrow \underset{x}{d}R\underset{z}{d}$

$eRd, dRe \rightarrow eRe$

$\begin{tikzcd} d \arrow[loop left] \end{tikzcd}$ transitive

$$\begin{tikzcd} & a & \\ b & & c \end{tikzcd}$$

R on $A = \{a, b, c\}$

irreflexive
symmetric
antisymmetric
transitive

} vacuous truths

Special combinations of properties:

① equivalence relation: reflexive, symmetric, transitive

② partial order: reflexive, antisymmetric, transitive

example: let R be a relation on $(\mathbb{Z}^+)^2$ ordered pairs of positive int.

such that $(a,b) \, R \, (c,d)$ iff $ad = bc$.

e.g. $(5,2) \, R \, (25,10)$ since $5 \cdot 10 = 2 \cdot 25$

$50 = 50$ ✓

Prove R is an equivalence relation.

① reflexive: $ab = ba$, so by def'n of R,

$(a,b) \, R \, (a,b)$ for all $a,b \in \mathbb{Z}^+$

② symmetric: let (a,b) and $(c,d) \in (\mathbb{Z}^+)^2$,

and suppose $(a,b) \, R \, (c,d)$.

Then, by def'n of R, $ad = bc \Rightarrow cb = da$.

So, by def'n of R $(c,d) \, R \, (a,b)$.

So, R symmetric.

③ transitive: let $(a,b), (c,d), (e,f) \in (\mathbb{Z}^+)^2$,

and suppose $(a,b) \, R \, (c,d)$, $(c,d) \, R \, (e,f)$.

Then $ad = bc$ and $cf = ed$.

$c = \dfrac{ed}{f}$

$$a\cancel{d} = \frac{be\cancel{d}}{f} \qquad d \neq 0$$

$$af = be \rightarrow (a,b)\,R\,(e,f)$$

Yes, transitive.

* please look at textbook for antisymmetry proof *